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Feynman's Relativistic Chessboard as an Ising Model

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Feynman has described a chessboard model for a one-dimensional relativistic quantum problem which yields the correct kernel for a free spin-1/2 particle moving in one spatial dimension. This chessboard problem can be solved as an Ising model, using the transfer matrix technique of statistical mechanics. The 2×2 transfer matrix represents the infinitesimal time evolution operator for the two eigenstates of the velocity operator.

Feynman and Hibbs (1965) have described a chessboard model for a one-dimensional relativistic quantum problem which yields the correct kernel for a free spin-1/2 particle moving in one spatial dimension. In this model, the particle's motion is restricted to be either forward or backward at the velocity of light. In the system of units for which $\hbar=m=c=1$, trajectories of such a particle in the z, t plane are straight lines with slopes of $\pm45°$ (bishop's moves in chess) as shown in Figure 1. The probability amplitude K for starting at $z_a t_a$ and ending at $z_b t_b$ is defined by dividing time into equal steps of length ε, with path reversals supposed to occur only at the boundaries of the steps, that is, at times $t=t_a+n\varepsilon$, n being an integer. The amplitude ϕ to go from $z_a t_a$ to $z_b t_b$ along one jagged path with R corners such as shown in Figure 1 is defined as

$$\phi=(-i\varepsilon)^R \tag{1}$$

which differs from that given in Feynman and Hibbs (1965) only by replacing Feynman's $i\varepsilon$ by $-i\varepsilon$. This replacement is apparently necessary to get the correct nonrelativistic limit [see remarks after equation (35)].

The kernel K is obtained by multiplying the probability amplitude ϕ for a path with R corners by $N(R)$, the number of paths possible with R

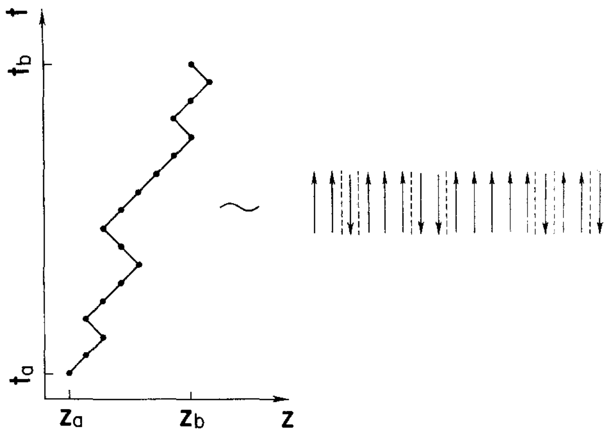

Fig. 1. A relativistic particle trajectory on the chessboard and the equivalent Ising spin system. The number of path segments equals the number of spins, $N=17$, with M_+, the number of paths to the right equal to the number of $+$ spins equal to 12, and M_-, the number of paths to the left equal to the number of $-$ spins equal to 5. The number of corners in the trajectory is $R=7$, the same as the number of $(+-)$ spin boundaries.

corners,

$$K(z_b - z_a, t_b - t_a) = \sum_R N(R)(-i\varepsilon)^R \tag{2}$$

The evaluation of K, left as a problem for the reader of Feynman and Hibbs (1965), apparently involves the combinatorial problem of enumerating values $N(R)$. Although solvable, calculating K from such a counting scheme is a bit tedious, and not very illuminating.

It is the purpose of this paper to show that the kernel K of equation (2) can be calculated analytically in closed form by establishing a correspondence betwen the above chessboard problem and the one-dimensional Ising model. Results for K obtained in this way are shown to be equivalent to the Dirac equation.

Involved in the chessboard problem is a total number of steps N determined by the time interval $t_b - t_a$ and the length ε of each step,

$$N\varepsilon = t_b - t_a \tag{3}$$

If N_+ (N_-) are the number of steps to the right (left), $N = N_+ + N_-$, then their difference, $M = N_+ - N_-$ is determined by the space interval $z_b = z_a$,

$$z_b - z_a = M\varepsilon \tag{4}$$

Now consider a system of N spins, $\sigma_1, \sigma_2, \ldots, \sigma_N$, each of which can take on the values ± 1. Associate σ_i with the ith chessboard step, $\sigma_i = +1$ denoting a step to the right and $\sigma_i = -1$ a step to the left. Then $M = N_+ - N_-$ in the chessboard problem becomes the magnetization $M = \Sigma_1^N \sigma_i$ for the spin problem. The total number of spins N, determined according to equation (3), by the time interval $t_b - t_a$ in units of the basic length ε is a large number, while the magnetization M determined by the interval $z_b - z_a$, given by equation (4), is again a large number. A corner in the jagged path of a particle in the zt plane corresponds to a boundary between oppositely directed neighboring spins. This equivalence is indicated in Figure 1, for a specific choice of M and N.

For a given spin configuration, the number of corners R (number of adjacent $\pm$spins) is measured by

$$R = (1/2) \sum_{1}^{N-1} (1 - \sigma_i \sigma_{i+1}) \tag{5}$$

We can manufacture a partition function $\mathfrak{Z}(N, M, j)$ for the spin system in thermal equilibrium which is exactly equivalent to the kernel K of equation (2) by assuming a nearest-neighbor ferromagnetic energy J so that every spin reversal boundary (corner) has the Boltzmann factor e^{-2j}, $j = J/kT$, with the zero of energy taken as the fully aligned ($M = N$) ferromagnetic state ($R = 0$). Then the partition function for N spins with fixed magnetization M is

$$\mathfrak{Z}(N, M, j) = \sum_{R} N(R)(e^{-2j})^R \tag{6}$$

with $N(R)$, the number of spin configurations having R oppositely aligned adjacent spins, the same combinatorial number as in equation (2) for the chessboard problem. The partition function $\mathfrak{Z}$ of equation (6) is identical to the kernel K of equation (2) if we set (finally) $e^{-2j} = -i\varepsilon$, and recall $N = (t_b = t_a)/\varepsilon$, $M = (z_b - z_a)/\varepsilon$. An alternative, potentially more useful form of $\mathfrak{Z}$ is obtained by recognizing that summing $N(R)$ over all R for fixed M gives all points $(\sigma_1, \sigma_2, \ldots, \sigma_N)$, $\sigma_i = \pm 1$ which lie on the plane $\Sigma_1^N \sigma_i = M$, or $\Sigma_R N(R) = \Sigma_{\sigma_1 = \pm 1} \cdots \Sigma_{\sigma_N = \pm 1}(1)$. With $\Sigma \sigma_i = M$ then replacing R in equation (6) by equation (5), the partition function can also be written as

$$\mathfrak{Z}(N, M, j) = \sum_{\sigma_1 = \pm 1} \sum_{\sigma_2 = \pm 1} \cdots \sum_{\sigma_N = \pm 1} e^{j \sum_{1}^{N-1} (\sigma_i \sigma_{i+1} - 1)} \tag{7}$$

$$\sum_{1}^{N} \sigma_i = M$$

which transfers the counting problem of evaluating $N(R)$ into the planar restriction $\Sigma_1^N \sigma_i = M$ on allowable spin configurations. Although $\mathfrak{Z}(N, M, j)$ could be evaluated in this form, it is much simpler to use the partition function $Z(N, \mu, j)$ for an Ising system with fixed external magnetic field $H = \mu kT$, rather than the $\mathfrak{Z}$ of equation (7) where the magnetization M is fixed. The two partition functions are related to each other in this way

$$Z(N, \mu, j) = \sum_{M=-N}^{N} e^{M\mu} \mathfrak{Z}(N, M, j) \tag{8}$$

$$= \sum_{\sigma_1 = \pm 1} \sum_{\sigma_2 = \pm 1} \cdots \sum_{\sigma_N = \pm 1} \exp\left(\mu \sum_1^N \sigma_i\right) \exp\left[j \sum_1^{N-1} (\sigma_i \sigma_{i+1} - 1)\right] \tag{9}$$

with the unconstrained sums over the σ_i in Z much easier to calculate than the constrained sums in $\mathfrak{Z}$. Whereas $\mathfrak{Z}(N, M, j)$ corresponds to the kernel $K(z, t)(z = z_b - z_a, t = t_b - t_a)$ in space-time, $Z(N, \mu, j)$ corresponds to the same free particle kernel expressed in momentum-time. To see this recall that in the chessboard problem, the usual integral relation connecting $K(z, t)$ and $K(p, t)$ is to be replaced by a sum over discrete steps of $z_b - z_a = M\mu$,

$$K(p, t) = \sum_{z_b - z_a} e^{-ip(z_b - z_a)} K(z_b - z_a, t) \tag{10}$$

Comparing equations (8) and (10), one sees the correspondence

$$Z(N, \mu, j) \rightarrow K(p, t) \tag{11}$$

with

$$M\mu = -ip(z_b - z_a) \tag{12}$$

or, since $M = (z_b - z_a)/\varepsilon$,

$$\mu = -ip\varepsilon \tag{13}$$

Now in the chessboard problem, $K(z_b - z_a, t_b - t_a)$ is the sum of four separate K's: K_{++}, K_{--}, K_{-+}. The first of these, K_{++}, is the probability amplitude for starting at $z_a t_a$ with positive velocity $+c$ and ending on $z_b t_b$ with positive velocity $+c$, the remainder of the K's defined in a similar fashion. To each of these K's in space-time, there corresponds, according to equation (10), an amplitude in momentum-time, e.g., $K_{++}(z, t) \rightarrow K_{++}(p, t)$. These latter four kernels can be separately calculated from the Ising model by leaving unspecified the spins σ_1 and σ_N in equation (8), thereby calculating $Z_{\sigma_1\sigma_N}(N, \mu, j)$,

$$Z_{\sigma_1\sigma_N}(N,\mu,j)=\sum_{\sigma_2=\pm 1}\cdots\sum_{\sigma_{N-1}=\pm 1}\exp\left[\mu\sum_1^N\sigma_i+j\sum_1^{N-1}(\sigma_i\sigma_{i+1}-1)\right] \tag{14}$$

Kramers and Wannier (1941) first showed how to calculate $Z_{\sigma_1\sigma_N}$ in closed form, using the transfer matrix technique. Let the 2×2 matrix $\mathcal{H}$ be defined by

$$\mathcal{H}(\sigma_i,\sigma_{i+1})=\exp\left[\tfrac{1}{2}\mu(\sigma_i+\sigma_{i+1})+j(\sigma_i\sigma_{i+1}-1)\right] \tag{15}$$

Then equation (14) becomes[1]

$$Z_{\sigma_1\sigma_N}=\sum_{\sigma_2=\pm 1}\cdots\sum_{\sigma_{N-1}=\pm 1}\mathcal{H}(\sigma_1\sigma_2)\mathcal{H}(\sigma_2\sigma_3)\cdots\mathcal{H}(\sigma_{N-1}\sigma_N) \tag{16}$$

Diagonalizing the matrix $\mathcal{H}(\sigma_1\sigma_2)$ yields two eigenvalues $\lambda_\pm$ and corresponding eigenvectors $\Phi_\pm(\sigma_1), \Phi_\pm(\sigma_2)$ which are orthogonal and assumed normalized,

$$\sum_\sigma \Phi_i(\sigma)\Phi_j(\sigma)=\delta_{ij} \tag{17}$$

In terms of these, the matrix elements of the transfer matrix $\mathcal{H}$ are

$$\mathcal{H}(\sigma_1\sigma_2)=\lambda_+\Phi_+(\sigma_1)\Phi_+(\sigma_2)+\lambda_-\Phi_-(\sigma_1)\Phi_-(\sigma_2) \tag{18}$$

and a sum over the intermediate spin σ_2 in the product of the first two matrices gives the result

$$\sum_{\sigma_2=\pm 1}\mathcal{H}(\sigma_1\sigma_2)\mathcal{H}(\sigma_2\sigma_3)=\lambda_+^2\Phi_+(\sigma_1)\Phi_+(\sigma_3)+\lambda_-^2\Phi_-(\sigma_1)\Phi_-(\sigma_3) \tag{19}$$

Therefore, summing on all spins from 2 to $N-1$ in equation (16) yields

$$Z_{\sigma_1\sigma_N}=\lambda_-^N\Phi_-(\sigma_1)\Phi_-(\sigma_N)+\lambda_+^N\Phi_+(\sigma_1)\Phi_+(\Phi_N) \tag{20}$$

[1]Equation (16), to be precisely correct, should contain the factor $\exp[\mu(\sigma_1+\sigma_N)/2]$. However, since this term goes to unity in the limit $\mu\rightarrow -ip\varepsilon$, $\varepsilon\rightarrow 0$, it is omitted here.

Eigenvalues $\lambda_{\pm}$ are solutions to

$$\begin{vmatrix} e^{\mu}-\lambda & e^{-2j} \\ e^{-2j} & e^{-\mu}-\lambda \end{vmatrix}=0 \tag{21}$$

Applied to the chessboard problem, $e^{-2j}\to -i\varepsilon$, $\mu\to -ip\varepsilon$, and, anticipating the final limit $\varepsilon\to 0$, it is sufficient to set $e^{\mu}\to 1-ip\varepsilon$, giving the matrix elements for $\mathcal{H}(\sigma_1,\sigma_2)$,

$$\mathcal{H}=\begin{array}{ccc} & (+) & (-) \\ (+) & 1-ip\varepsilon & -i\hat{\varepsilon} \\ (-) & -i\varepsilon & 1+ip\varepsilon \end{array} \tag{22}$$

The off-diagonal elements $-i\varepsilon$ correspond to the corners in Figure 1, and introduce the phase $=-i\varepsilon$ as given in equation (1). Diagonal elements, corresponding to no cornering, introduce the phases $1\mp ip\varepsilon$ for $\pm c$ trajectories. The eigenvalues λ of $\mathcal{H}(\sigma_1\sigma_2)$ are

$$\lambda_{+}=1\pm i\varepsilon E \tag{23}$$

with

$$E=(1+p^2)^{1/2} \tag{24}$$

the relativistic free particle energy in the system of units $m=c=1$. Eigenvectors for λ_{+} are found to be

$$\Phi_{+}(\sigma=+1)=\left[\tfrac{1}{2}(1-p/E)\right]^{1/2} \tag{25}$$

$$\Phi_{+}(\sigma=-1)=-\left[\tfrac{1}{2}(1+p/E)\right]^{1/2} \tag{26}$$

and those for λ_{-} are given by

$$\Phi_{-}(\sigma=+1)=\left[\tfrac{1}{2}(1+p/E)\right]^{1/2} \tag{27}$$

$$\Phi_{-}(\sigma=-1)=\left[\tfrac{1}{2}(1-p/E)\right]^{1/2} \tag{28}$$

The eigenvector $\Phi_{+}(\sigma=+1)$ is the probability amplitude at any time, or at any step along the t axis, $t=t_a+n\varepsilon$ for the particle to have the velocity $+c$ in the eigenstate $\lambda_{+}=1+i\varepsilon E$. Similarly $\Phi_{+}(\sigma=-1)$ is the probability amplitude at any time for the particle to have the velocity $-c$ in the eigenstate λ_+. For $p=0$, all four probabilities obtained from equations (25)–(28) are seen to be equal to 1/2. This is consistent with the behavior of the Ising model spins in zero magnetic field μ [$p=0$ implies $\mu=0$ from equation (13)], when an up spin must be as likely as a down spin. Using these eigenvalues and eigenvectors in equation (20) for Z_{σ_1,σ_N}, and setting $N\to t/\varepsilon$, $\mu\to -ip\varepsilon$, $(1\pm i\varepsilon E)^{t/\varepsilon}\to e^{\pm iEt}$, we find

$$K_{++}(p,t)=\tfrac{1}{2}(1+p/E)e^{-iEt}+\tfrac{1}{2}(1-p/E)e^{iEt} \tag{29}$$

$$K_{--}(p,t)=\tfrac{1}{2}(1-p/E)e^{-iEt}+\tfrac{1}{2}(1+p/E)e^{iEt} \tag{30}$$

$$K_{-+}(p,t)=(1/2E)e^{-iEt}-(1/2E)e^{iEt} \tag{31}$$

$$=K_{+-}(p,t) \tag{32}$$

the last relation being obvious from right–left symmetry. Note that in contrast to the standard thermal problem in which eigenvalues $\lambda_\pm$ are real, so that one need take only the larger root in the limit $N\to\infty$, in the present case both roots contribute, $(\lambda_\pm)^N=e^{\pm iEt}$, one ($\lambda_-$) giving positive energy solutions, the other (λ_+) negative energy solutions. Inspection of equations (29)–(32) shows that negative energy solutions are obtained from those for positive energy by replacing E by $-E$, so that the kernels K_{ij} could be neatly expressed as a 2×2 matrix, which for positive energy has diagonal elements $\frac{1}{2}(1\pm p/E)$ and $1/2E$ for off-diagonal elements.

In the high-energy limit, $p\gg 1$, $E\to p$, and $K_{++}(p,t)\to e^{-ipt}$, $K_{--}(p,t)\to e^{ipt}$ and $K_{+-}\to 0$. This limit corresponds to vanishing off-diagonal elements of the transfer matrix in equation (22). Since it is these which produce corners in the chessboard model, only straight line trajectories $z=\pm t$ result for $E=p$, as is evident from

$$K_{++}(z,t)=\int\frac{dp}{2\pi}e^{ipz}e^{-ipt}=\delta(z-t)\quad\text{and}\quad K_{--}(z,t)=\delta(z+t).$$

The corresponding Ising analogy of this situation is N spins with such strong interaction j ($e^{-2j}\to 0$) that if the first spin is $+$, all the remainder are also $+$, or if the first spin is $-$, the remainder are also $-$.

The constants $\hbar, m, c$ may be reinstated by replacing t in the above relations by $t/(\hbar/mc^2)$ and p by p/mc. Then $E=[1+p^2/(mc^2)]^{1/2}$ is the energy in units of mc^2. The kernels in space-time, $K(z,t)$ can be obtained from the $K(p,t)$ given above by Fourier integration. The sum of all four kernels is

$$K(p,t)=(1+1/E)e^{-iEt}+(1-1/E)e^{iEt} \tag{33}$$

In the nonrelativistic limit, $E=(1+p^2)^{1/2}\approx 1+p^2/2$, and in this limit $K(p,t)$ tends to

$$K(p,t)\approx 2e^{-it}e^{-ip^2t/2} \tag{34}$$

or, in ordinary units

$$K(p,t)\approx 2e^{-imc^2t/\hbar}e^{-ip^2t/2m\hbar} \tag{35}$$

Except for the factor $\exp(-imc^2t/\hbar)$, this kernel $K(p,t)$ [and its Fourier transform $K(z,t)$] is correct for a nonrelativistic particle. Had we chosen the phase factor introduced at a corner, equation (1), to be $i\varepsilon$ rather than $-i\varepsilon$, the four kernels would be the same as given in equations (29)–(32), except for K_{+-} (and K_{-+}), which would be the negative of that given in equation (32). Then the nonrelativistic limit would come out as in equation (35) above, but with the opposite signs in the exponents.

EQUIVALENCE WITH THE DIRAC EQUATION

To show that the momentum space kernels given by equations (29)–(32) are the same as those obtained from the Dirac equation, we consider that equation for a free particle with momentum p, energy E, in the system of units $\hbar=m=c=1$,

$$(\alpha_z p+\beta)\Psi(p)=E\Psi(p) \tag{36}$$

To conform to the chessboard problem we choose a representation in which the velocity operator $dz/dt=\alpha_z$ is diagonal with eigenvalues ± 1,

$$\alpha_z=\begin{pmatrix}\sigma_z & 0\\ 0 & -\sigma_z\end{pmatrix} \tag{37}$$

$$\beta=\begin{pmatrix}0 & -I\\ -I & 0\end{pmatrix} \tag{38}$$

Eigenstates of dz/dt in this chiral representation are

$$\alpha_z\Psi_{\pm}=\pm\Psi_{\pm} \tag{39}$$

or

$$\Psi_{+}=2^{-1/2}\begin{pmatrix}1\\ 0\\ 0\\ 1\end{pmatrix} \tag{40}$$

and $\Psi_- = \beta \Psi_+$,

$$\Psi_- = 2^{-1/2} \begin{pmatrix} 0 \\ -1 \\ -1 \\ 0 \end{pmatrix} \tag{41}$$

The states $\Psi_\pm$ correspond to the $\pm c$ paths in the chessboard problem. Starting at $t=0$ with the state $\Psi_+(0)$ ($+c$ trajectory), in a small time interval it evolves into $\Psi_+(\Delta t)$,

$$\Psi_+(\Delta t) = [1 - i(p\alpha_z + \beta)\Delta t]\Psi_+(0) \tag{42}$$

Using equations (36)–(41) this becomes

$$\Psi_+(\Delta t) = (1 - ip\Delta t)\Psi_+(0) - i\Delta t \Psi_-(0) \tag{43}$$

Alternatively, the state $\Psi_-(0)$ ($-c$ trajectory) evolves in time Δt into

$$\Psi_-(\Delta t) = (1 + ip\Delta t)\Psi_-(0) - i\Delta t \Psi_+(0) \tag{44}$$

For either choice of initial trajectory, putting $\Delta t = \varepsilon$,

$$\begin{pmatrix} \Psi_+(\varepsilon) \\ \Psi_-(\varepsilon) \end{pmatrix} = \begin{pmatrix} 1 - ip\varepsilon & -i\varepsilon \\ -i\varepsilon & 1 + ip\varepsilon \end{pmatrix} \begin{pmatrix} \Psi_+(0) \\ \Psi_-(0) \end{pmatrix} \tag{45}$$

Comparison of equation (45) and equation (22) shows that the initial states are transferred in time ε by the same transfer matrix as in the Ising model version of the chessboard problem. The off-diagonal matrix elements $-i\varepsilon$ in equation (42) are clearly due to the mass term β in the Dirac equation–eigenstates of α_z are not eigenstates of the energy since β anticommutes with α_z.

For a finite time interval t, $t = N\varepsilon$, N-fold iteration of the transfer matrix in equation (45) would define precisely the same problem treated earlier, and therefore yield the same kernels $K_{ij}(p, t)$ as previously found.

More conventionally, these kernels may be obtained by expanding $\psi_\pm$ in terms of energy-momentum eigenstates of the free particle in this chiral representation (Bjorken and Drell, 1964). Positive energy solutions, $E = (1 + p^2)^{1/2}$ are

$$\Psi_1 = N \begin{pmatrix} 1 \\ 0 \\ -1/E+p \\ 0 \end{pmatrix} \tag{46}$$

and

$$\Psi_2 = N \begin{pmatrix} 0 \\ -1/E+p \\ 0 \\ 1 \end{pmatrix} \tag{47}$$

while negative energy solutions are

$$\Psi_3 = N \begin{pmatrix} 1/E+p \\ 0 \\ 1 \\ 0 \end{pmatrix} \tag{48}$$

and

$$\Psi_4 = N \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1/E+p \end{pmatrix} \tag{49}$$

where $E = |E|$, and where the common factor N is given by

$$N = \left[\tfrac{1}{2}(1+p/E)\right]^{1/2} \tag{50}$$

To calculate one kernel, say $K_{+-}(p,t)$, we need the probability amplitude that a particle of momentum p, in the state Ψ_+ at $t=0$ will be found in the state Ψ_- at time t, $K_{+-}(p,t) = (\Psi_-(0), \Psi_+(t))$. To obtain this scalar product, both states $\Psi_\pm$ can be expanded in the free particle solutions,

$$\Psi_\pm(t) = (C_{1\pm}\Psi_1 + C_{2\pm}\Psi_2)e^{-iEt} + (C_{3\pm}\Psi_3 + C_{4\pm}\Psi_4)e^{iEt} \tag{51}$$

The coefficients are easily seen to be

$$C_{3+} = C_{4+} = \tfrac{1}{2}(1-p/E)^{1/2} \tag{52}$$

$$C_{3-} = C_{4-} = -\tfrac{1}{2}(1+p/E)^{1/2} \tag{53}$$

$$C_{1+} = C_{2+} = \tfrac{1}{2}(1+p/E)^{1/2} \tag{54}$$

$$C_{1-} = C_{2-} = \tfrac{1}{2}(1-p/E)^{1/2} \tag{55}$$

They are arranged in sequence so as to correlate with the four equations

(25)–(28) for the eigenvectors in the transfer matrix problem. Thus, for example C_{3+}, C_{4+} in equation (52) are probability amplitudes for a $+c$ trajectory in a negative energy state, identical in meaning and value (except for a normalization factor $2^{1/2}$) to the eigenvector $\Phi_+(\sigma=+)$ in equation (25).

The kernel $K_{+-}(p,t)$ expressed in terms of the C's is

$$K_{+-}(p,t)=(C_{1+}C_{1-}+C_{2+}C_{2-})e^{-iEt}+(C_{3+}C_{3-}+C_{4+}C_{4-})e^{iEt} \tag{56}$$

or

$$K_{+-}(p,t)=(1/2E)e^{-iEt}-(1/2E)e^{iEt} \tag{57}$$

the same as equation (31) calculated via the transfer matrix technique. The same correspondence is easily seen to hold for K_{++} and K_{--}.

CONCLUSION

Feynman's chessboard problem can be solved as an Ising model, using the transfer matrix technique of statistical mechanics. The 2×2 transfer matrix represents the infinitesimal time evolution operator for the two eigenstates of the velocity operator.

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